Casimir Momentum of a Chiral Molecule in a Magnetic Field

M. Donaire,[1,2,*] B.A. van Tiggelen,[1] and G.L.J.A. Rikken[2]

[1] *Université Grenoble 1/CNRS, LPMMC UMR 5493, B.P. 166, 38042 Grenoble, France*
[2] *LNCMI, UPR 3228 CNRS/INSA/UJF Grenoble 1/UPS, Toulouse & Grenoble, France*

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In a classical description, a neutral, polarizable object acquires a kinetic momentum when exposed to crossed electric and magnetic fields. In the presence of only a magnetic field no such momentum exists classically, although it is symmetry-allowed for an object lacking mirror symmetry. We perform a full QED calculation to show that the quantum vacuum coupled to a chiral molecule provides a kinetic "Casimir" momentum directed along the magnetic field, proportional to its rotatory power and the fine structure constant.

The notion that electromagnetic quantum vacuum fluctuations contain energy is widely accepted. When coupled to matter they give rise to Van-der-Waal forces between neutral objects and Lamb-shifts of energy levels in atoms [1]. The attraction between metallic plates is undoubtedly the most convincing proof for the existence of Casimir energy [2]. The empty quantum vacuum is Lorentz-invariant with a diverging energy density with frequency spectrum ω^3 and zero momentum. The regularization of this divergence is still an issue in the context of the cosmological constant problem [3]. In all other cases the UV catastrophe is not essential when binding energies and forces are calculated in a dielectric medium [4].

The possibility of the quantum vacuum to contribute to the momentum of matter was first raised in Ref. [5]. For a macroscopic object exposed to crossed electric and magnetic fields, $\mathbf{E}_0$ and $\mathbf{B}_0$ respectively, a QED correction was obtained in addition to the classical Abraham momentum density, $(\epsilon-1)(\mathbf{E}_0\wedge\mathbf{B}_0)$, which emerges from Maxwell's equations and is already controversial by itself as concerning the prefactor. Being an observable quantity, the UV divergence of the momentum obtained in Ref. [5] poses a real problem if only because the end result depends crucially on an heuristic cut-off. In Refs. [6, 7] two of us have shown that the divergence can be handled in a full quantum theory that goes beyond the dipole approximation. Divergencies were seen to disappear by mass regularization and a finite but small Casimir momentum remained that is a factor α^2 smaller than the classical effect. In the quantum theory it is also clear that two contributions exist to the (pseudo) momentum. One is the familiar sum $\sum_{\mathbf{k},\epsilon}\hbar\mathbf{k}(a^\dagger_{\mathbf{k}\epsilon}a_{\mathbf{k}\epsilon}+1/2)$ associated with photons with momentum $\hbar\mathbf{k}$, and often referred to as the "transverse" component [8]. The second more subtle "longitudinal" contribution $\sum_i q_i\mathbf{A}(\mathbf{r}_i)$ stems from the electromagnetic gauge field coupled to electric charges $\{q_i\}$ and was found to dominate the Casimir momentum.

In this work we apply the same formalism as in Refs. [6, 7] to address the Casimir momentum of a quantum object subject to only a magnetic field. The latter being a pseudo-vector, symmetry imposes the object to be chiral, i.e. we anticipate the relation $\mathbf{P}^{\rm Cas} = ge\mathbf{B}_0$ with g some pseudo-scalar quantifying the broken mirror symmetry of the quantum object. Important is that no classical contribution has been reported, so that the proposed contribution from the quantum vacuum would constitute 100% of the effect. Our theory is nonrelativistic, and gives a finite magneto-chiral Casimir momentum. We find g to be proportional to the product of the static rotatory power of the molecule and the fine structure constant.

We propose the simplest molecular model that exhibits all necessary features to leading order in perturbation theory: broken mirror symmetry, Zeeman splitting of energy levels and coupling to the quantum vacuum, and we neglect relativistic effects. In our model the optical activity of the molecule is determined by a single chromophoric electron within a chiral object which is further simplified to be a two-particle system in which the chromophoric electron of charge $q_e = -e$ and mass m_e is bound to a nucleus of effective charge $q_N = e$ and mass $m_N \gg m_e$. The binding interaction is modeled by a harmonic oscillator potential, $V^{HO} = \frac{\mu}{2}(\omega_x^2x^2+\omega_y^2y^2+\omega_z^2z^2)$, to which we add a term $V_C = C\,xyz$ to break the mirror symmetry. The coordinates x, y, z are those of the relative position vector, $\mathbf{r} = \mathbf{r}_N - \mathbf{r}_e$, and $\mu = \frac{m_Nm_e}{M}$ with $M = m_N + m_e$. The center of mass position vector is $\mathbf{R} = (m_N\mathbf{r}_N + m_e\mathbf{r}_e)/M$. V_C was firstly introduced by Condon *et al.* [9, 10] to explain rotatory power with a single oscillator model. The parameter C is a pseudoscalar which derives from the Coulomb interaction of the system with its chiral environment. When an external uniform and constant magnetic field $\mathbf{B}_0$ is applied the total Hamiltonian of the system reads, $H = H_0 + H_{EM} + W$,

$$\text{with } H_0 = \sum_{i=e,N}\frac{1}{2m_i}[\mathbf{p}_i - q_i\mathbf{A}_0(\mathbf{r}_i)]^2 + V^{HO} + V_C, \quad (1)$$

$$H_{EM} = \sum_{\mathbf{k},\epsilon}\hbar\omega_{\mathbf{k}}(a^\dagger_{\mathbf{k}\epsilon}a_{\mathbf{k}\epsilon} + \frac{1}{2}) + \frac{1}{2\mu_0}\int \mathrm{d}^3r\mathrm{B}_0^2, \quad (2)$$

$$W = \sum_{i=e,N}\frac{-q_i}{m_i}[\mathbf{p}_i - q_i\mathbf{A}_0(\mathbf{r}_i)]\cdot\mathbf{A}(\mathbf{r}_i) + \frac{q_i^2}{2m_i}\mathrm{A}^2(\mathbf{r}_i). \quad (3)$$

In the electromagnetic (EM) vector potential we have separated the contribution of the external classical field, $\mathbf{A}_0(\mathbf{r}_i) = \frac{1}{2}\mathbf{B}_0 \wedge \mathbf{r}_i$, from the one of the quantum field operator, $\mathbf{A}(\mathbf{r}_i)$. In the sum, $a^\dagger_{\mathbf{k}\epsilon}$ and $a_{\mathbf{k}\epsilon}$ are the creation and annihilation operators of photons with momentum $\hbar\mathbf{k}$, frequency $\omega_{\mathbf{k}} = ck$ and polarization vector ϵ respectively. The magnetostatic energy of the integral is a constant irrelevant to us that we will discard. The system obeying the above Hamiltonian posseses a conserved pseudo-momentum [6, 8],

$$\hat{\mathbf{K}} = \mathbf{P} + \frac{e}{2}\mathbf{B}_0 \wedge \mathbf{r} + \sum_{\mathbf{k},\epsilon} \hbar\mathbf{k}(a^\dagger_{\mathbf{k}\epsilon} a_{\mathbf{k}\epsilon} + \frac{1}{2}), \qquad (4)$$

which satisfies $[H, \hat{\mathbf{K}}] = \mathbf{0}$. Its eigenvalues are therefore good quantum numbers. $\hat{\mathbf{K}}$ differs from both the conjugate total momentum, $\mathbf{P} = \mathbf{p}_e + \mathbf{p}_N$, and the kinetic momentum, $\mathbf{P}_{\rm kin} = M\dot{\mathbf{R}} = \mathbf{P} - \frac{e}{2}\mathbf{B}_0 \wedge \mathbf{r} - e[\mathbf{A}(\mathbf{r}_N) - \mathbf{A}(\mathbf{r}_e)]$. The terms in $\hat{\mathbf{K}}$ can be also arranged as,

$$\hat{\mathbf{K}} = \mathbf{P}_{\rm kin} + \mathbf{P}_{\rm Abr} + \mathbf{P}^{\rm Cas}_{\parallel} + \mathbf{P}^{\rm Cas}_{\perp}, \qquad (5)$$

where, beside the kinetic momentum, $\mathbf{P}_{\rm Abr} = e\mathbf{B}_0 \wedge \mathbf{r}$ is the Abraham momentum and we define the Casimir momentum, $\mathbf{P}^{\rm Cas} = \mathbf{P}^{\rm Cas}_{\parallel} + \mathbf{P}^{\rm Cas}_{\perp}$, as the momentum of the vacuum field. $\mathbf{P}^{\rm Cas}$ is composed of a longitudinal Casimir momentum, $\mathbf{P}^{\rm Cas}_{\parallel} = e[\mathbf{A}(\mathbf{r}_N) - \mathbf{A}(\mathbf{r}_e)]$, and a transverse Casimir momentum, $\mathbf{P}^{\rm Cas}_{\perp} = \sum_{\mathbf{k},\epsilon} \hbar\mathbf{k}(a^\dagger_{\mathbf{k}\epsilon} a_{\mathbf{k}\epsilon} + \frac{1}{2})$ [8]. Our aim is to evaluate $\mathbf{P}^{\rm Cas}$ in the ground state of our system. The Casimir momentum, we will show, is equivalent in magnitude to the kinetic momentum acquired by the chiral oscillator.

In the first stage of the calculation we neglect the vacuum field. The conserved pseudo-momentum of the resultant Hamiltonian, H_0, is $\hat{\mathbf{Q}} = \mathbf{P}_{\rm kin} + e\mathbf{B}_0 \wedge \mathbf{r}$, with $[H_0, \hat{\mathbf{Q}}] = \mathbf{0}$ and continuous eigenvalues $\mathbf{Q}$. The unitary operator $\mathrm{U} = \exp[i(\mathbf{Q} - \frac{e}{2}\mathbf{B}_0 \wedge \mathbf{r}) \cdot \mathbf{R}/\hbar]$ maps the Hamiltonian H_0 into $\tilde{H}_0 = \mathrm{U}^\dagger H_0 \mathrm{U}$, which conveniently separates internal and external motion,

$$\tilde{H}_0 = \frac{1}{2M}\mathrm{Q}^2 + \frac{1}{2\mu}\mathrm{p}^2 + V^{HO} + V_C + V_Z + \mathcal{O}(\mathrm{B}_0^2). \quad (6)$$

In this equation $\mathbf{p}$ is the conjugate momentum of $\mathbf{r}$, $\mathbf{p} = \mu(\mathbf{p}_N/m_N - \mathbf{p}_e/m_e)$, and $V_Z = \frac{e}{2\mu^*}(\mathbf{r} \wedge \mathbf{p}) \cdot \mathbf{B}_0$ is the Zeeman potential with $\mu^* = \frac{m_N m_e}{m_N - m_e}$.

The ground state of the Hamiltonian $\tilde{H}_0$ is, up to order $C\mathrm{B}_0$ [11],

$$\begin{aligned}
|\tilde{\Omega}_0\rangle &= |0\rangle - \mathcal{C}|111\rangle - i\mathcal{B}_0^z \eta^{yx}|110\rangle \\
&+ i\mathcal{B}_0^z \mathcal{C}\eta^{yx}\left(|001\rangle + 2|221\rangle\right) \\
&- \sqrt{2}i\mathcal{B}_0^z\mathcal{C}\left(\frac{2\omega_x - \omega_z\eta^{yx}}{\omega_z + 2\omega_x}|201\rangle - \frac{2\omega_y + \omega_z\eta^{yx}}{\omega_z + 2\omega_y}|021\rangle\right) \\
&+ \sum \text{ cyclic permutations.}
\end{aligned}$$

Correspondingly, the ground state of H_0 is $|\Omega_0\rangle = \mathrm{U}|\tilde{\Omega}_0\rangle$, with a pseudo-momentum $\mathbf{Q}_0$ to be fixed. The fact that $\langle\Omega_0|\mathbf{r}|\Omega_0\rangle = \mathbf{0}$ implies also that $\mathbf{Q}_0$ is the kinetic momentum of the oscillator in absence of vacuum. In the above equation the states $|n_x n_y n_z\rangle$ refer to the eigenstates of the harmonic oscillator Hamiltonian. The dimensionless parameters are, $\mathcal{B}_0^i = \frac{eB_0^i}{4\mu^*\sqrt{\omega_j\omega_k}}$ with $i \neq j \neq k$ and $i \neq k$, $\mathcal{C} = \frac{C\hbar^{1/2}}{(2\mu)^{3/2}(\omega_x+\omega_y+\omega_z)(\omega_x\omega_y\omega_z)^{1/2}}$, $\eta^{ij} = \frac{\omega_i - \omega_j}{\omega_i + \omega_j}$. Here, the indices i, j, k take on the three spatial directions, x, y, z. The η factors are assumed to be small quantities which quantify the anisotropy of the oscillator. They *all* have to be nonzero for the optical activity of the molecule to survive rotational averaging. In the following, all our calculations restrict to the lowest order in $\mathcal{C}$, $\mathcal{B}_0^i$ and η^{ij}.

Next, we couple the oscillator to the vacuum field and evaluate the Casimir momentum, $\langle\mathbf{P}^{\rm Cas}\rangle = \langle\Omega|\mathbf{P}^{\rm Cas}|\Omega\rangle$. Here, $|\Omega\rangle$ is the ground state of the oscillator when coupled to the vacuum field. It will be computed applying up to second order perturbation theory to $|\Omega_0\rangle$ with the interaction potential W. In general, $\langle\mathbf{P}^{\rm Cas}\rangle$ will depend on the pseudo-momentum of $|\Omega_0\rangle$, $\mathbf{Q}_0$. In the presence of the vacuum, $\mathbf{Q}_0$ is not the kinetic momentum of the system but the quantum number of the operator $\hat{\mathbf{K}}$ instead. It can be shown that $\langle\mathbf{P}\rangle = \mathbf{Q}_0 - \frac{e}{2}\mathbf{B}_0 \wedge \langle\mathbf{r}\rangle - \langle\mathbf{P}^{\rm Cas}_{\perp}\rangle$. By inserting this relation into Eq.(4) we find, $\langle\hat{\mathbf{K}}\rangle = \mathbf{Q}_0$. For simplicity we take $\mathbf{Q}_0 = \mathbf{0}$, which corresponds to an oscillator that would be at rest in absence of vacuum, and evluate the expectation value of the operators in Eq.(5). The induced dipole moment, $\langle e\mathbf{r}\rangle$, vanishes, and so does $\langle\mathbf{P}_{\rm Abr}\rangle$. The rest of the terms yield, $\langle\mathbf{P}_{\rm kin}\rangle + \langle\mathbf{P}^{\rm Cas}_{\parallel}\rangle + \langle\mathbf{P}^{\rm Cas}_{\perp}\rangle = \mathbf{0}$, for $\mathbf{Q}_0 = \mathbf{0}$. This means that a nonzero value of the Casimir momentum implies a nonzero kinetic momentum in the molecule entirely due to its interaction with the vacuum field. The transverse momentum, $\langle\mathbf{P}^{\rm Cas}_{\perp}\rangle = \sum_{\mathbf{k},\epsilon} \hbar\mathbf{k}\langle\Omega|a^\dagger_{\mathbf{k}\epsilon} a_{\mathbf{k}\epsilon}|\Omega\rangle$, is found to be finite but small. The longitudinal momentum, $\langle\mathbf{P}^{\rm Cas}_{\parallel}\rangle = e\langle\Omega|\mathbf{A}(\mathbf{r}_N) - \mathbf{A}(\mathbf{r}_e)|\Omega\rangle$, gives the dominant contribution that we will evaluate in detail first.

At $\mathcal{O}(C\mathrm{B}_0)$ and lowest order in the coupling constant e, it suffices to compute $|\Omega\rangle$ applying first order perturbation theory to $|\Omega_0\rangle$. We use the U-transformed states and the U-transformed potential,

$$\begin{aligned}
\tilde{W} &= -\frac{e}{m_N}\left(\mathbf{p} + \frac{m_N}{M}\mathbf{P} - \frac{e}{2}\mathbf{B}_0 \wedge \mathbf{r}\right) \cdot \mathbf{A}(\mathbf{R} + \frac{m_e}{M}\mathbf{r}) \\
&- \frac{e}{m_e}\left(\mathbf{p} - \frac{m_e}{M}\mathbf{P} + \frac{e}{2}\mathbf{B}_0 \wedge \mathbf{r}\right) \cdot \mathbf{A}(\mathbf{R} - \frac{m_N}{M}\mathbf{r}) \\
&+ \frac{e^2}{2m_N}\mathrm{A}^2(\mathbf{R} + \frac{m_e}{M}\mathbf{r}) + \frac{e^2}{2m_e}\mathrm{A}^2(\mathbf{R} - \frac{m_N}{M}\mathbf{r}), \quad (7)
\end{aligned}$$

to arrive at,

$$\langle\mathbf{P}^{\rm Cas}_{\parallel}\rangle = \sum_{\mathbf{Q},I,\gamma_{\mathbf{k}\epsilon}} \frac{\langle\tilde{\Omega}_0|e\Delta\mathbf{A}|\mathbf{Q},I,\gamma\rangle\langle\mathbf{Q},I,\gamma|\tilde{W}|\tilde{\Omega}_0\rangle}{E_0 - E_{Q,I,k}} + c.c., \qquad (8)$$

where $\Delta\mathbf{A} = \mathbf{A}(\mathbf{r}_N) - \mathbf{A}(\mathbf{r}_e)$ and $E_0 = \hbar(\omega_x+\omega_y+\omega_z)/2$. The intermediate atomic states, I, are eigenstates of $\tilde{H}_0$ and may have a priori any pseudo-momentum $\mathbf{Q}$. Zeros in the denominator are avoided in the summation. It turns out that only the terms proportional to $\mathbf{p}\cdot\mathbf{A}$ in $\tilde{W}$ yield a nonvanishing contribution to $\langle\mathbf{P}_\parallel^{\rm Cas}\rangle$. Writing the EM quantum field in Eq.(8) as usual [12],

$$\mathbf{A}(\mathbf{r}) = \sum_{\mathbf{k},\epsilon}\sqrt{\frac{\hbar}{2ck\mathcal{V}\epsilon_0}}[\epsilon a_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}} + \epsilon^* a_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}}], \quad (9)$$

with $\mathcal{V}$ a generic volume, c the speed of light and ϵ_0 the vacuum permittivity, passing the sums over $\mathbf{Q}$ and $\mathbf{k}$ to continuum integrals and by summing over polarization states we arrive at,

$$\begin{aligned}\langle\mathbf{P}_\parallel^{\rm Cas}\rangle &= \frac{\hbar e^2}{2c\epsilon_0}\int\frac{{\rm d}^3k}{(2\pi)^3k}\langle\tilde{\Omega}_0|(e^{i\frac{m_e}{M}\mathbf{k}\cdot\mathbf{r}} - e^{-i\frac{m_N}{M}\mathbf{k}\cdot\mathbf{r}})\\ &\times \frac{(\mathbb{I}-\frac{\mathbf{k}\otimes\mathbf{k}}{k^2})\cdot}{\hbar^2k^2/2M + \hbar ck - E_0 + \tilde{H}_0}\\ &\times \left(\frac{e^{-i\frac{m_e}{M}\mathbf{k}\cdot\mathbf{r}}}{m_N} + \frac{e^{i\frac{m_N}{M}\mathbf{k}\cdot\mathbf{r}}}{m_e}\right)\mathbf{p}|\tilde{\Omega}_0\rangle + c.c., \quad (10)\end{aligned}$$

where $\mathbb{I}$ is the unit dyadic and the kinetic energy $\hbar^2k^2/2M$ in the denominator is a consequence of the momentum recoil of photon to atom, $\mathbf{Q} = -\mathbf{k}$. When the oscillatory exponentials on the left of the Hamiltonian operator are moved to the right, two types of term appear. To the first type belong those terms in which the products of the oscillatory exponentials cancel. They correspond to processes in which a photon is created and annihilated at the same particle (i.e., electron or nucleus). The second class contains terms in which the product of the exponentials amount to $e^{\pm i\mathbf{k}\cdot\mathbf{r}}$. They correspond to processes in which a photon is created and annihilated at different particles. The first kind of terms are the dominant ones. The second kind of terms are a factor $\sqrt{E_0/\mu c^2}$ smaller and will be neglected.

The dominant terms are,

$$\begin{aligned}\langle\mathbf{P}_\parallel^{\rm Cas}\rangle &= \Re\Big\{\Big[\frac{-\hbar e^2}{3\pi^2c\epsilon_0 m_e}\int_0^\infty\frac{k{\rm d}k}{\frac{\hbar^2k^2}{2m_e}+\hbar ck}\langle\tilde{\Omega}_0|\mathbf{p}|\tilde{\Omega}_0\rangle\\ &+ \frac{\hbar e^2}{3\pi^2c\epsilon_0 m_e}\int_0^\infty\langle\tilde{\Omega}_0|\frac{k{\rm d}k}{\frac{\hbar^2k^2}{2m_e}+\hbar ck - E_0 + H^{HO}}\\ &\times (V_C+V_Z)\frac{1}{\frac{\hbar^2k^2}{2m_e}+\hbar ck - E_0 + H^{HO}}\mathbf{p}|\tilde{\Omega}_0\rangle\\ &- \frac{\hbar e^2}{3\pi^2c\epsilon_0 m_e}\int_0^\infty\langle 0|\frac{k{\rm d}k}{\frac{\hbar^2k^2}{2m_e}+\hbar ck - E_0 + H^{HO}}\\ &\times (V_C+V_Z)\frac{1}{\frac{\hbar^2k^2}{2m_e}+\hbar ck - E_0 + H^{HO}}\\ &\times (V_Z+V_C)\frac{1}{\frac{\hbar^2k^2}{2m_e}+\hbar ck - E_0 + H^{HO}}\mathbf{p}|0\rangle\Big]\\ &- [m_e\leftrightarrow m_N]\Big\}, \quad (11)\end{aligned}$$

where $-[m_e \leftrightarrow m_N]$ means that the same expression within the square brackets must be evaluated exchanging m_e with m_N and subtracted. $H^{HO} = {\rm p}^2/2\mu + V^{HO}$ is the harmonic ocillator Hamiltonian. Note that, since both the chiral interaction V_C and the Zeeman effect V_Z feauturing in $\tilde{H}_0$ are treated as perturbations to the harmonic oscillator, the denominator in Eq.(10) is expanded up to order V_CV_Z in Eq.(11). Hence, only terms at $\mathcal{O}(C{\rm B}_0)$ must be retained. When moving the oscillatory exponentials from the left to the right of the Hamiltonian operator in Eq.(10), in application of canonical commutation relations the momentum $\mathbf{p}$ in $\tilde{H}_0$ gets shifted, $\mathbf{p} \to \mathbf{p} \pm \frac{m_{N,e}}{M}\hbar\mathbf{k}$. As a result, the kinetic energy in the denominators of Eq.(11) becomes $\hbar^2k^2/2m_{e,N}$ and an additional term $\pm\hbar\mathbf{k}\cdot\mathbf{p}/m_{e,N}$ shows up there. The latter is a Doppler shift term which has been neglected in Eq.(11).

In Eq.(11), the factor which accompanies $\langle\tilde{\Omega}_0|\mathbf{p}|\tilde{\Omega}_0\rangle$ is a UV divergent integral. It was shown in Refs. [6, 7] that it can be interpreted as a mass renormalization factor. In our case this divergence does not have any effect on the Casimir momentum since $\langle\tilde{\Omega}_0|\mathbf{p}|\tilde{\Omega}_0\rangle = \mathbf{0}$. All the other integrals in Eq.(11) contain more than one factor quadratic in k in their denominators, which makes them fully convergent. When integrating in k up to ∞ the leading contribution of them all is logarithmic. The end result can be expressed as,

$$\langle\mathbf{P}_\parallel^{\rm Cas}\rangle = \frac{Ce^3\ln(m_N/m_e)}{96\pi^2c\epsilon_0\mu\mu^*(\omega_x+\omega_y+\omega_z)}\sum_{i,j,k}\varepsilon_{ijk}\frac{{\rm B}_0^i\eta^{kj}}{\omega_k\omega_j}\hat{i}, \quad (12)$$

where ε_{ijk} is the three-dimensional Levi-Civita tensor. Note that $\eta^{kj} = -\eta^{jk}$. We thus need the anisotropy of the oscillator quantified by η^{jk} to make the Casimir momentum nonzero. The momentum is still not averaged over the random orientation of the chiral object. To do so we must project $\langle\mathbf{P}^{\rm Cas}\rangle$ onto the magnetic field vector and extract the trace,

$$\langle\mathbf{P}_\parallel^{\rm Cas}\rangle_{rot} = \frac{Ce^3\ln(m_e/m_N)\mathbf{B}_0}{144\pi^2c\epsilon_0\mu\mu^*\omega_x\omega_y\omega_z}\eta^{zy}\eta^{xz}\eta^{yx}, \quad (13)$$

where the subscript *rot* denotes rotational average. The logarithmic dependence on Eqs.(12,13) may be disputed since it results from wave numbers larger than $m_{N,e}c/\hbar$. It is possible that relativistic corrections are needed, which lie outside the scope of the present Letter.

The transverse Casimir momentum, $\langle\mathbf{P}_\perp^{\rm Cas}\rangle$, corresponds to the expectation value of the operators in the sum over modes of Eq.(5). Its evaluation involves a second order pertubation calculation whose details will be published somewhere else [11]. The end result is approximately,

$$\begin{aligned}\langle\mathbf{P}_\perp^{\rm Cas}\rangle_{rot} &= \frac{-0.98Ce^3\mathbf{B}_0}{144\pi^2c\epsilon_0 m_e^2\omega_x\omega_y\omega_z}\eta^{zy}\eta^{xz}\eta^{yx} \quad (14)\\ &\simeq \langle\mathbf{P}_\parallel^{\rm Cas}\rangle_{rot}/\ln(m_N/m_e).\end{aligned}$$

For instance, if we take for m_N the mass of a carbon atom, $\langle \mathbf{P}_\perp^{\rm Cas}\rangle_{rot}$ is an order of magnitude less than $\langle \mathbf{P}_\parallel^{\rm Cas}\rangle_{rot}$.

Eq.(13) and Eq.(14) give simple formulas for $\langle \mathbf{P}_\parallel^{\rm Cas}\rangle_{rot}$ and $\langle \mathbf{P}_\perp^{\rm Cas}\rangle_{rot}$ in terms of the chiral parameter C, the magnetic field and the different oscillator frequencies. We will now express them in terms of the static optical rotatory power and electric polarizability of the molecule, which can be easily calculated for this simple model and which are both directly observable. To this aim we derive equations for the electric and magnetic dipole moments induced in the state $|\Omega_0\rangle$ of the molecule by external electric and magnetic fields of frequency ω, $\mathbf{E}_\omega$ and $\mathbf{B}_\omega$ respectively. For this we apply first order perturbation theory to $|\tilde{\Omega}_0\rangle$ using the potential $\tilde{W}$ in which the quantum field must be substituted by a classical EM vector potential from which the external electric and magnetic fields derive. After performing the rotational average we can write for the molecule's electric and magnetic dipole moments, respectively,

$$\begin{aligned}\langle \mathbf{d}\rangle_{rot} &= \alpha_E \mathbf{E}_\omega + \gamma\, \mathbf{B}_0 \wedge \mathbf{B}_\omega - \beta\, \dot{\mathbf{B}}_\omega, \qquad (15)\\ \langle \mathbf{m}\rangle_{rot} &= \alpha_M \mathbf{B}_\omega + \zeta\, \mathbf{B}_0 \wedge \dot{\mathbf{B}}_\omega + \gamma\, \mathbf{B}_0 \wedge \mathbf{E}_\omega + \beta\, \dot{\mathbf{E}}_\omega,\end{aligned}$$

where we have adopted Condon's notation [9] in SI units. In this equation α_E and α_M are the electric and magnetic polarizabilities respectively. ζ describes the magnetic Faraday effect, β is the molecular rotatory factor responsible for the natural optical activity and γ gives rise to the magneto-chiral anisotropy. In agreement with Ref. [13] they all behave as $\sim \omega^{-2}$ for $\omega \gg E_0/\hbar$ whereas they tend to constant values in the static limit, $\omega \to 0$, where we find,

$$\beta(0)/\alpha_E(0) = \frac{\hbar C \mathcal{D}}{8\mu\mu^* \omega_x\omega_y\omega_z}\eta^{zy}\eta^{xz}\eta^{yx}. \qquad (16)$$

In this formula $\mathcal{D}$ is a rational polynomial of the natural frequencies of the oscillator which approximates $\mathcal{D} \approx 1$ for not too large anisotropy. By comparing the above expression with Eqs.(13,14) we conclude that the total Casimir momentum, $\langle \mathbf{P}^{\rm Cas}\rangle_{rot} = \langle \mathbf{P}_\parallel^{\rm Cas}\rangle_{rot} + \langle \mathbf{P}_\perp^{\rm Cas}\rangle_{rot}$, can be written approximately as,

$$\langle \mathbf{P}^{\rm Cas}\rangle_{rot} = \frac{-2\alpha}{9\pi}\frac{\beta(0)}{\alpha_E(0)}[\ln(m_N/m_e)+1]e\mathbf{B}_0, \qquad (17)$$

where α is the fine structure constant. Thus, we confirm our conjecture, $\mathbf{P}^{\rm Cas} = ge\mathbf{B}_0$, with $g = \frac{-2\alpha}{9\pi}\frac{\beta(0)}{\alpha_E(0)}[\ln(m_N/m_e)+1]$. We speculate that, apart from constants of order unity, this expression is model-independent.

The presence of $\hbar$ in Eq.(16) deserves a comment. The fact that β is proportional to $\hbar$ is a consequence of the quantum formalism. For a given set of natural frequencies, all the atomic lengths in the problem are determined by quantum mechanics. In particular, $\beta(0)/\alpha_E(0)$ is a length, a fraction of the electronic Compton wavelength, that we identify with a chiral length, l_{ch}. Therefore, we can write $\mathrm{P}^{\rm Cas} \sim \alpha\mu l_{ch}\omega_C$, where $\omega_C = e\mathrm{B}_0/\mu$ is the cyclotron frequency.

Next, we compare our result with earlier work on Casimir momentum wich is based on the application of the fluctuation-dissipation theorem and worked out in the dipole approximation [14, 15]. For our model that approach would yield,

$$\mathbf{P}^{\rm Cas} = \frac{\hbar\mathbf{B}_0}{3\pi^2\epsilon_0 c}\Re\int_0^{k_{max}} \mathrm{d}k\, k^3\gamma(k). \qquad (18)$$

This expression is analogous to the formula obtained in Refs. [5, 16] for the transverse Casimir momentum transferred to a homogeneous magnetoelectric medium. In contrast to the quantum calculation, the integral of Eq.(18) diverges quadratically for $k_{max} \to \infty$ and needs a UV cut-off to yield a finite result. Upon comparing the result of this integral with the quantum result of Eq.(14) we have identified this cut-off with the inverse Compton wavelength, $k_{max} \sim \frac{\mu c}{\hbar}$. The details of the calculation will be published elsewhere [11].

Finally, we estimate the orders of magnitude of the Casimir momentum transferred to a single molecule of an actual chiral compound. To this aim we derive values for $\beta(0)$ and $\alpha_E(0)$ from experimental parameters. Let us consider a molecular medium with a number density ρ. At leading order in ρ and far from resonances the rotatory power of the medium, φ, relates to the molecular rotatory factor β as, $\varphi = 4\pi^2\rho\beta(0)/\lambda^2\epsilon_0$, and the refractive index is $n = 1 + \rho\alpha_E(0)/2\epsilon_0$. As an example we take the chiral compound 2-octanol, $C_8H_{18}O$. According to Table I of Ref. [9], at $\lambda = 4800$Å, $n = 1.43$ and its specific rotatory power is $[\varphi]^{4800}$=15.46 deg/dm/(g/cm^3). Its mass density is 0.82 g/cm^3 and its molecular mass is 130.2 u. We find, $\beta(0) = 3\times10^{-53}$radC^{2}m^3/J, $\alpha_E(0) = 2\times10^{-39}$m^{2}C^2/J. Considering that the chiral center is one of the carbon atoms and using $\mathrm{B}_0 = 10$T, we find $\langle \mathrm{P}^{\rm Cas}\rangle_{rot} = 1.4\times10^{-34}\,$kg m/s. This corresponds to an average velocity of 0.6 nm/s approximately, which lies within the scope of current experimental observations [17, 18].

In this Letter we have carried out a quantum computation of the Casimir momentum transfered from the quantum vacuum to a chiral oscillator subject to an external magnetic field. The result is free of divergences, linear in $\mathbf{B}_0$ and proportional to α, which proves the genuine QED nature of the effect. Our calculation is based on a simplified molecular model which allows us to express the final result in terms of observable parameters, i.e., the molecular rotatory factor and the electric polarizability. Eq.(17) is the main result of this Letter. A quantitative prediction has been made whose verification is accessible to current experiments.

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* Electronic address: manuel.donaire@grenoble.cnrs.fr

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